11. Calculus

Differentiation

You should be able to:

• understand the basic informal principles of limits and convergence
• find the derivative of a polynomial function from first principles
• use a rule to differentiate algebraic functions containing sums and differences of \( kx^n \), \( n \in \mathbb{Q} \)
• differentiate the functions \( \sin x, \cos x, \tan x, \ln x \) and \( e^x \)
• find the second derivative of a function
• interpret the derivative as the gradient function
• interpret the derivative as a rate of change.

You should know:

• the notation \( f'(x) \) or \( \frac{dy}{dx} \) is used to denote the first derivative of a function. The second derivative is denoted by \( f''(x) \) or \( \frac{d^2y}{dx^2} \)

• derivatives of
  \[ \frac{d}{dx} kx^n = nkx^{n-1} \]
  \[ \frac{d}{dx} \sin x = \cos x \]
  \[ \frac{d}{dx} \cos x = -\sin x \]
  \[ \frac{d}{dx} \tan x = \frac{1}{\cos^2 x} \]
  \[ \frac{d}{dx} \ln x = \frac{1}{x} \text{ and } \frac{d}{dx} e^x = e^x \]

• the gradient function can be used to find the gradient of the tangent to the curve at a point and hence the gradient of the curve at that point.

Example

Let \( f(x) = x^3 - 2x^2 - 1 \).

(a) Find \( f'(x) \).

Applying the power rule for derivatives of polynomials, we have \( f'(x) = 3x^2 - 4x \).

(b) Find the gradient of the curve of \( f(x) \) at the point \((2, -1)\).

We are asked to find the gradient of the curve when \( x = 2 \), that is, \( f'(2) \).

\[ f'(2) = 3(2)^2 - 4(2) = 4 \]

Therefore, the gradient of the curve (and hence, the gradient of the tangent) at \( x = 2 \) is 4.

Be prepared

• Work carefully when finding the derivative from first principles as it is easy to get lost in the algebra.

• Write polynomial functions in an appropriate form before attempting to find the derivative. For example, the function \( f(x) = 6\sqrt{x^2} \) should be expressed as \( f(x) = 6x^{2/3} \) first.
Tangents and normals

You should be able to:
- use the derivative to find the equation of a tangent or a normal to a graph
- use the equation of a tangent or a normal to a point on a graph to find the coordinates of that point.

You should know:
- the tangent to a curve at a point is the straight line that touches the curve at that point but does not cross it
- the tangent to the graph of \( f \) at the point where \( x = a \) has equation \( y - f(a) = f'(a)(x - a) \)
- the normal to a curve is the line perpendicular to the tangent at a point on the curve
- the normal to the curve at \( x = a \) has gradient \(-\frac{1}{f'(a)}\)

Example
Consider the function \( f(x) = 3x^2 - 5x + k \).
(a) Write down \( f'(x) \).
\[
f'(x) = 6x - 5
\]

(b) The equation of the tangent to the graph of \( f \) at \( x = p \) is \( y = 7x - 9 \).

(i) Find the value of \( p \).
The gradient of the tangent line is 7. Setting \( f'(x) = 7 \), we have
\[
6x - 5 = 7, \text{ so } x = 2
\]
Hence the value of \( p \) is 2.

(ii) Find the value of \( k \).
To find the value of \( k \), we must first find the corresponding \( y \)-value on the graph of \( f \) when \( x = 2 \).
As the point also lies on the tangent, it satisfies the equation. Hence,
\[
y = 7(2) - 9 = 5
\]
Consequently, the point \( (2, 5) \) lies on the graph of \( f \), so, substituting into the original function,
\[
5 = 3(2)^2 - 5(2) + k
k = 3
\]

Be prepared
- Be prepared to work backwards—that is, find the coordinates of a point on a graph if you know the gradient of the tangent or normal to that point.
Product rule, quotient rule and chain rule

You should be able to:

- use the product rule to differentiate functions of the form \( y = uv \) where \( u \) and \( v \) are both functions of \( x \)
- use the quotient rule to differentiate functions of the form \( y = \frac{u}{v} \)
- use the chain rule to differentiate composite functions of the form \( y = g(f(x)) \).

You should know:

- if \( y = uv \), and if both \( u \) and \( v \) are differentiable functions of \( x \), then
  \[
  \frac{dy}{dx} = u \frac{dv}{dx} + v \frac{du}{dx}
  \]
- if \( y = \frac{u}{v} \) then
  \[
  \frac{dy}{dx} = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}
  \]
- it is useful to think of the chain rule as the “outside–inside” rule. First, differentiate the “outside” function, evaluate it at the “inside” function, then multiply the result by the derivative of the “inside” function. That is, if \( y = g(f(x)) \), then
  \[
  \frac{dy}{dx} = g'(f(x)) \times f'(x).
  \]

Example

Differentiate each of the following with respect to \( x \).

(a) \( y = \sin 3x \)

This is a composite function made up of the functions \( \sin x \) and \( 3x \). Applying the chain rule, we have

\[
\frac{dy}{dx} = \sin(3x) \times \frac{d}{dx} 3x = 3 \cos 3x
\]

(b) \( y = x \tan x \)

The function \( y \) is the product of two functions \( x \) and \( \tan x \). It is helpful to find the derivative of each of these functions first, then substitute correctly into the formula for the product rule.

\[
\frac{d}{dx} x = 1 \quad \text{and} \quad \frac{d}{dx} \tan x = -\frac{1}{\cos^2 x}
\]

Substituting,

\[
\frac{d}{dx} x \tan x = x \frac{d}{dx} \tan x + \tan x \frac{d}{dx} x
\]

\[
= x \frac{\tan x}{\cos^2 x} + \tan x
\]

(c) \( y = \frac{\ln x}{x} \)

This question is asking us to find the derivative of the quotient of two functions \( \ln x \) and \( x \). We know that

\[
\frac{d}{dx} \ln x = \frac{1}{x} \quad \text{and that} \quad \frac{d}{dx} x = 1.
\]

Substituting into the quotient rule, we have

\[
\frac{dy}{dx} = \frac{x \frac{d}{dx} \ln x - \ln x \frac{d}{dx} x}{x^2}
\]

\[
= \frac{x \frac{1}{x} - \ln x}{x^2}
\]

\[
= \frac{1 - \ln x}{x^2}
\]

Be prepared

- The key to successful differentiation is in recognizing which rule to apply.
- The phrase “rate of change” suggests that you need to find a derivative.
- You may need to use a combination of rules when finding the derivative of a function.
Graphical interpretations of the derivative

You should be able to:
- use derivatives to analyse the shape and key features of a graph, including points of inflexion, maxima and minima and concavity
- interpret and analyse the relationships between the graphs of $f$, $f'$ and $f''$.

You should know:
The following graphs can be used to help you interpret the statements below.

- the graph of $f$ is increasing on an interval if $f'(x) > 0$ and decreasing if $f'(x) < 0$ for every $x$ in the interval
- $f''(x)$ is the rate of change of $f'(x)$ in the same way that $f'(x)$ is the rate of change of $f(x)$
- if $x = c$, $f'(c) = 0$
- the graph of $f$ has a minimum value if $f'$ changes from negative to positive at $c$
- the graph of $f$ has a maximum value if $f'$ changes from positive to negative at $c$
- the graph of $f$ has no extreme values if $f'$ has the same sign on both sides of $c$
- the graph of $f$ is concave upwards in an interval if the tangent lines drawn at every point in the interval are increasing in gradient (Figure 1). Consequently, if $f'(x)$ is increasing, then $f''(x) > 0$ for every $x$ in that interval
- the graph of $f$ is concave downwards in an interval if the tangent lines drawn at every point in the interval are decreasing in gradient (Figure 2). Consequently, if $f'(x)$ is decreasing, then $f''(x) < 0$ for every $x$ in that interval

Example
The graph of a function $g$ is given in the diagram below.

The gradient function $g''(x)$ has its maximum value at point B and its minimum value at point D. The tangent is horizontal at points C and E.

(a) Complete the table below, by stating whether the first derivative $g'$ is positive or negative, and whether the second derivative $g''$ is positive or negative.

<table>
<thead>
<tr>
<th>Interval</th>
<th>$g'$</th>
<th>$g''$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a &lt; x &lt; b$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$e &lt; x &lt; f$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
In the interval $a < x < b$, the graph of $g$ is increasing, so $g'$ is positive. The graph of $g$ is also concave upwards, so $g''$ is also positive.

In the interval $c < x < f$, the graph of $g$ is decreasing, so $g'$ is negative. The graph of $g$ is concave downwards in this interval, so $g''$ must be negative.

(b) Complete the table below by noting the points on the graph described by the following conditions.

<table>
<thead>
<tr>
<th>Conditions</th>
<th>Point</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g'(x) = 0$, $g''(x) &lt; 0$</td>
<td></td>
</tr>
<tr>
<td>$g'(x) &lt; 0$, $g''(x) = 0$</td>
<td></td>
</tr>
</tbody>
</table>

$g'(x) = 0$ implies a horizontal tangent line on the graph of $g$. This occurs at the points $C$ and $E$. However, at $C$, the graph is concave downwards, so $g''(x) < 0$, and at point $E$, $g''(x) = 0$ as it is also a point of inflexion. Hence, the point $C$ is the only point where $g'(x) = 0$ and $g''(x) < 0$.

We are told that the gradient function has a minimum value at the point $D$, so $g''(x) = 0$. Also, at $D$, the graph of $g(x)$ is decreasing so that $g'(x) < 0$. Hence, the point $D$ satisfies both of the given criteria.

**Be prepared**

- If there is a point of inflexion, then $f''(x) = 0$. But the converse is not necessarily true. Knowing $f''(x) = 0$ does not guarantee a point of inflexion.
- For a point of inflexion to exist at $x = c$, you must show not only that $f''(c) = 0$ but also that the sign of $f''$ changes on either side of $c$. 
Optimization

You should be able to:
- find the maximum or minimum value of a function
- use the first or second derivative to justify whether a function has a local maximum or a local minimum
- apply the concept of maximum and minimum to problems for which there is an optimal solution.

You should know:
- to find the maximum or minimum value of a function, set \( f'(x) = 0 \)
- a function has a maximum value at \( x = c \) if \( f'(c) = 0 \) and \( f'(x) \) changes from positive to negative on either side of \( c \)
- a function has a minimum value at \( x = c \) if \( f'(c) = 0 \) and \( f'(x) \) changes from negative to positive on either side of \( c \)
- the second derivative can also be used to justify a maximum or a minimum value. If \( f''(c) < 0 \), then \( f \) has a local maximum at \( x = c \). If \( f''(c) > 0 \), then \( f \) has a local minimum at \( x = c \)
- the derivative can be used in applications where maximum and minimum values are required. These are called optimization problems. Examples include maximizing profit or volume and minimizing costs or surface area.

Example
The following diagram shows a rectangular area ABCD enclosed on three sides by 60 m of fencing, and on the fourth by a wall AB.

Find the width AD of the rectangle that gives it the maximum area and justify your answer.
Integration

You should be able to:

- understand that integration is the reverse of the process of differentiation
- integrate algebraic functions containing sums and differences of \( kn^x, \ n \in \mathbb{Q} \)
- integrate the functions \( \sin x, \cos x, \frac{1}{x} \) and \( e^x \), and composites of these functions with the linear function \( ax + b \)
- integrate \( f'(x) \) to find a function \( f(x) \) and use a given point (called a boundary condition) to find the constant term.

You should know:

- the basic integration rules are obtained by simply looking at integration as the "reverse process" of differentiation
  - \( \int kx^n \, dx = k \frac{x^{n+1}}{n+1} + C, \ n \neq -1 \)
  - \( \int \sin x \, dx = -\cos x + C \)
  - \( \int \cos x \, dx = \sin x + C \)
  - \( \int \frac{1}{x} \, dx = \ln x + C \)
  - \( \int e^x \, dx = e^x + C \)
- the solution to an equation that contains a derivative such as \( f'(x) = 3x^2 \) or \( \frac{dy}{dx} = \sin(2x - 1) \) is a function of the form \( y = f(x) + C \). A boundary condition can then be substituted to find a specific value of \( C \).

Example

If \( f'(x) = \cos x \), and \( f\left(\frac{\pi}{2}\right) = -2 \), find \( f(x) \).

We are given an equation involving a derivative, \( f'(x) = \cos x \), and a boundary condition, \( f\left(\frac{\pi}{2}\right) = -2 \).

We are asked to find a function \( f(x) \). So,

\[
 f(x) = \int \cos x \, dx = \sin x + C
\]

Substituting the boundary condition to find \( C \), we have

\[
 f\left(\frac{\pi}{2}\right) = \sin \frac{\pi}{2} + C = -2, \text{ so } 1 + C = -2,
\]

therefore \( C = -3 \) and \( f(x) = \sin x - 3 \)

Be prepared

- Write a function in an appropriate form before trying to find the integral. For example, \( \int x^2 + \frac{3}{x^3} \, dx \) is best written as \( \int (x + 3x^{-3}) \, dx \) using the laws of exponents and is then easily integrated as \( \frac{1}{2}x^2 - 3x^{-2} + C \).
- Include the constant of integration \( C \) when integrating to find a function \( f(x) \).
Area and the definite integral

You should be able to:

- evaluate a definite integral
- write and evaluate an expression to find the area under a curve
- find the area between two curves
- use a GDC to evaluate definite integrals and areas.

You should know:

- to evaluate a definite integral, use the following rule: \( \int_a^b f(x) \, dx = F(b) - F(a) \), where \( F \) is the anti-derivative of \( f(x) \)
- the properties of the definite integral are best explained using an example. Suppose that \( \int_1^3 f(x) \, dx = 5 \),\n  \[ \int_1^3 f(x) \, dx = \int_1^2 f(x) \, dx - \int_2^3 f(x) \, dx \]
  \[ \int_1^2 f(x) \, dx = 3 \] \[ \int_2^3 f(x) \, dx = -3 \]
  \[ \int_1^3 f(x) \, dx = 3 + (-3) = 2 \]
- \( \int_0^3 [3f(x) + 2g(x)] \, dx = 3 \int_0^3 f(x) \, dx + 2 \int_0^3 g(x) \, dx = 3(5) + 2(7) = 29 \)
- if a function is continuous and positive on an interval \( a \leq x \leq b \), then the area under the curve is given by the definite integral \( \int_a^b f(x) \, dx \)
- if a function lies below the x-axis on an interval \( a \leq x \leq b \), then the area under the curve is given by \( -\int_a^b f(x) \, dx \) or \( \int_b^a f(x) \, dx \)
- the area between two curves \( f(x) \) and \( g(x) \) is given by \( \int_a^b [f(x) - g(x)] \, dx \) where \( f(x) > g(x) \) on an interval \( a \leq x \leq b \).

The lower and upper limits of integration \( a \) and \( b \) are often but not always the x-coordinates where the two graphs intersect.

Example

The following diagram shows part of the graph of \( y = \cos x \) for \( 0 \leq x \leq 2\pi \). Regions A and B are shaded.

(a) Write down an expression for the area of A.

The region marked A lies completely above the x-axis from \( x = \frac{3\pi}{2} \) to \( x = 2\pi \). Therefore, the definite integral \( \int_{\frac{3\pi}{2}}^{2\pi} \cos x \, dx \) represents the area of A.

(b) Calculate the area of A.

Analytical solution

Integrating the expression in A and applying the rule for finding a definite integral, we have

\[
\int_{\frac{3\pi}{2}}^{2\pi} \cos x \, dx = \left[ \sin x \right]_{\frac{3\pi}{2}}^{2\pi}
\]

\[
= \sin 2\pi - \sin \frac{3\pi}{2}
\]

\[
= 0 - (-1)
\]

\[
= 1
\]

GDC solution

Entering the expression from part (a) into the GDC, we have:

- **Texas Instruments**
  \[ \text{fnInt} (\cos(x), X, 3, \pi/2, 2\pi) \]
  \[ 1 \]

- **Casio**
  \[ \int (\cos X, 3\pi/2, 2\pi) \]
  \[ 1 \]

or from the graphing screen:

- **Texas Instruments**
  \[ \text{Area} \]
  \[ 1 \]

- **Casio**
  \[ \text{Area} \]
  \[ 1 \]

Hence, the area of region A is 1.
Area and the definite integral (continued)

(c) (i) Given that $\sin \frac{4\pi}{3} = -\frac{\sqrt{3}}{2}$ find the area of
region B.

Region B lies below the x-axis. We are asked for the total
area so we require the area of region B to be positive. This
can be achieved by switching the limits on our definite
integral so $\int_{3\pi/2}^{\pi/3} \cos x \, dx$ represents the area of region B.

Area of B = $\int_{3\pi/2}^{\pi/3} \cos x \, dx$

= $[\sin x]_{3\pi/2}^{\pi/3}$

= $[\sin \frac{4\pi}{3} - \sin \frac{3\pi}{2}]$

= $-\frac{\sqrt{3}}{2} - (-1)$

= $1 - \frac{\sqrt{3}}{2}$

(ii) Hence, find the total area of the shaded
regions.

The total area is then given by

$\int_{\pi/3}^{\pi/2} \cos x \, dx + \int_{3\pi/2}^{\pi/3} \cos x \, dx$

$= 1 + 1 - \frac{\sqrt{3}}{2}$

$= 2 - \frac{\sqrt{3}}{2}$

The GDC may be used as in (b) to find the total, inexact
area of 1.13.

Be prepared

- Use more than one expression to find the area under a
curve that lies partially above and below the x-axis.
- Enter the expression for area into your GDC correctly.
- Indicate the required area on a sketch.
- Find a limit of integration if you are given the area.
Volumes of revolution

You should be able to:

- calculate the volume of a solid of revolution formed when a region under a graph is rotated 360° or 2π radians around the x-axis
- use the integral features of your GDC to calculate the volume.

You should know:

- when you fully rotate the area beneath the curve around the x-axis, a solid of positive volume is formed
- to find the volume of this solid, use the formula $V = \int_a^b \pi y^2 \, dx$, where $a$ and $b$ are the x-intercepts of the function.

Example

Consider the function $f(x) = e^{2x-1} + \frac{5}{2x-1}$, $x \neq \frac{1}{2}$. The region between the curve and the x-axis between $x = 1$ and $x = 1.5$ is rotated through 360° about the x-axis. Let $V$ be the volume formed.

(a) Write down an expression to represent $V$.

Using the formula for volume, we have

$$V = \pi \int_{a}^{b} \left(e^{2x-1} + \frac{5}{2x-1}\right)^2 \, dx$$

(b) Hence, write down the value of $V$.

The command term "write down" suggests that we do not use an analytical approach but rather a GDC. First enter the function into the equation editor. Then use the integral feature to obtain the correct answer.

Be prepared

- First set up an expression that represents the volume even if the question does not ask for it.
- Don't forget to square the function when entering the data into the GDC.
- Find an intercept/limit if a volume is given.
**Kinematics**

**You should be able to:**
- obtain the velocity and acceleration functions, $v(t)$ and $a(t)$, from the displacement function $s(t)$
- obtain the velocity and the displacement functions from the acceleration function
- interpret the area under a velocity–time graph as distance travelled.

**You should know:**
- displacement measures the change of position of an object as a function of time
- velocity is the derivative of the displacement function, that is, $v(t) = \frac{ds}{dt}$
- acceleration is the derivative of velocity with respect to time or the second derivative of position, that is $a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}$
- if $v(t) = 0$, the object is **not moving** and has reached its maximum or minimum height or distance
- if $a(t) = 0$, the object’s velocity is constant and the object may have reached its maximum or minimum velocity
- $\int a(t) \, dt = v(t) + C$ and $\int v(t) \, dt = s(t) + C$
- if the graph of $v(t)$ is continuous and positive on an interval $a \leq t \leq b$, then the area under the curve is given by the definite integral $\int_a^b v(t) \, dt$ and represents the distance travelled by an object during that interval of time.

**Example**
A car starts moving from a fixed point A. Its velocity $v$ metres per second after $t$ seconds is given by $v = 4t + 5 - 5e^{-t}$. Let $d$ be the displacement from A when $t = 4$.

(a) Write down an integral that represents $d$.

The velocity function is positive and increasing over the first 4 seconds. This can be determined by looking at a graph of $v$ or recognizing that, as $t$ increases, so does $v$. Hence, the area under the curve is equal to the displacement from A. Therefore,

$$d = \int_0^4 (4t + 5 - 5e^{-t}) \, dt$$

(b) Find the value of $d$.

**Analytical approach**

Integrating, we have

$$\int (4t + 5 - 5e^{-t}) \, dt = \left[ 2t^2 + 5t - 5e^{-t} \right]_0^4$$

$$= (32 + 20 + 5e^{-4}) - (5)$$

$$= 47 + 5e^{-4}$$

**GDC approach**

Using the integral function in graphing mode, we have:

*Texas Instruments*

*Casio*

Hence, the displacement from A is about 47.1 metres.